A high-order regularization method and approximate solution

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1 Introduction

A range of sensor network problems, such as localization problems [1, 2, 3], can be described using linear models; even when the original models are nonlinear, their linearized approximation can still lead to meaningful solutions. Consider a set of linear equations

$$Ax = b \tag{1}$$

where $x \in \mathbf{R}^n$, $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$. The least square (LS) method and its variations are often used in solving such linear problems (1). However, it is more challenging when the matrix A is ill-conditioned ([4], Chapter 5.8); in this case, an approximate solution given by the Tikhonov regularization (TR) method [5] can be handy. A standard form of the TR solution can be found in [5], which is given by

$$x_{TR} = \left(A^TA + \lambda^2 I\right)^{-1} A^T b \tag{2}$$

where λ is a given regularization parameter. It is known, though, the TR solution can be overly smooth without a proper justification of adding a regularization term to the original problem (1).

2 The high-order regularization method

To find a relationship between the LS solution and the TR solution, and get a better explainable solution to the original problem (1), we propose a high-order regularization (HR) method that is described as follows

$$x_{HR} = \left(A^TA + R\right)^{-1} \sum_{l=0}^{k} \left(R\left(A^TA + R\right)^{-1}\right)^l A^T b \tag{3}$$

where the regularization matrix $R \in \mathrm{S}_+^n$ is a diagonal matrix and the matrix $\left(A^TA + R\right)$ is non-singular. When $k = 0$, the HR solution is equivalent to the TR solution with $R = \lambda^2 I$, and the condition $\rho\left(R\left(A^TA + R\right)^{-1}\right) < 1$ required in the matrix series can always be satisfied. In general, we can select $k = 1$, and for a diagonal regularization matrix, the HR solution can be simplified as

$$x_{HR} = \left(A^TA + R\right)^{-1} A^T b + \left(A^TA + R\right)^{-1} R \left(A^TA + R\right)^{-1} A^T b \tag{4}$$

which is a better solution than the TR solution in the sense of approximation of the inverse of the matrix A^TA.

3 Example

A mobile robot localization problem can be formulated into a matrix form like (1) with

$$A = \begin{bmatrix} (p_1 - p_{m+1})^T \\ \vdots \\ (p_m - p_{m+1})^T \end{bmatrix}, \; b = \begin{bmatrix} p_1^T p_1 - p_{m+1}^T p_{m+1} + d_{m+1}^2 - d_1^2 \\ \vdots \\ p_m^T p_m - p_{m+1}^T p_{m+1} + d_{m+1}^2 - d_m^2 \end{bmatrix} \tag{5}$$

where $A \in \mathbf{R}^{m\times 3}$, $p_i = (x_i, y_i, z_i)^T$, $i = 1, \ldots, m+1$ are positions of $m+1$ anchors with known given positions and d_i are distance measurements between the i-th anchor and the robot. Therefore, the position estimation of the robot can be given by the high-order regularization method (3).

4 Conclusion

In this work, we proposed a high-order regularization method to solve the linear sensor localization problem when the associated matrix is ill-conditioned. We show that the proposed method is superior to the Tikhonov regularization in approximating the inverse problem.

References

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